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by Alexander Mendelson and Samuel W. Spero

Lewis Research Center
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SUMMARY

A method is presented for obtaining directly the elastic stress distribution in a thin, finite-width orthotropic plate with a central crack. The solution, which was obtained by the use of finite Fourier transforms, involves the solving of an integral equation for the crack opening; the stresses can then be computed by simple quadrature. The solution is valid as long as the plate width is at least twice the crack length. It is shown that the stress intensity factor is independent of the orthotropy of the material.

INTRODUCTION

The elastic distribution of stresses and strains around a crack in a finite or an infinite medium is a problem basic to the expanding field of fracture mechanics. The solution for the isotropic infinite plate has been obtained by several methods and is well known. The solution for an isotropic finite plate with a central crack has only been obtained approximately (ref. 1) by using Westergaard's semi-inverse solution for colinear cracks (ref. 2). The problem of the orthotropic infinite plate has been solved both for tension and for bending (ref. 3).

The present report presents a direct solution for an orthotropic finite-width plate with a central crack under tension. The solution is not exact in that the normal stress distribution on the free sides of the plate is not zero, although the resultant of this stress distribution is zero. It is shown that for the case of isotropy the solution agrees with that given by Westergaard's semi-inverse method for colinear cracks. In the limiting case, as the ratio of crack length to plate width approaches zero, the solution approaches the one for the infinite plate.

In the first part of this report the equations for the stress and strain distributions in an orthotropic plate are presented together with a discussion of their validity and limitations. In appendix A, this solution is derived in detail, in a purely formal manner, by using finite Fourier transforms.

SOLUTION

Consider a thin orthotropic plate 2 units in width with a central crack of length 2a, along the x-axis, loaded with a uniform tensile load P per unit

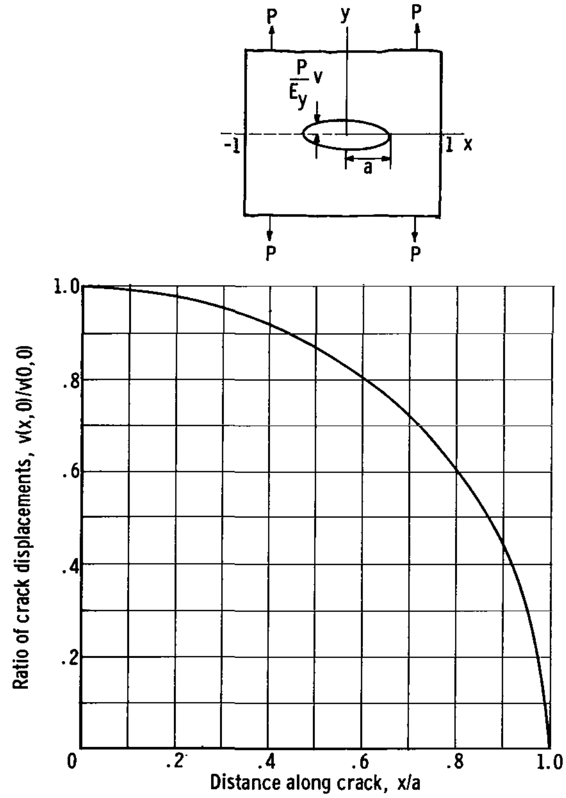

Figure 1. - Variation of crack opening with distance along crack for crack half-lengths from 0.001 to 0.5.

width at y equal to infinity, as shown in the sketch in figure 1. It is shown in appendix A that a formal solution for the stresses can be obtained in the following form:

$$\left.\begin{aligned}
S_y(x,y) &= 1 + \frac{2\pi}{\beta^2 - 1}\int_0^a v(\nu,0)\left[I_1(x,y,\nu) - \beta I_1\left(x,\frac{y}{\beta},\nu\right)\right]d\nu \\
S_x(x,y) &= \frac{2\pi}{\beta(\beta^2 - 1)}\int_0^a v(\nu,0)\left[I_1\left(x,\frac{y}{\beta},\nu\right) - \beta I_1(x,y,\nu)\right]d\nu \\
S_{xy}(x,y) &= -\frac{2\pi}{\beta^2 - 1}\int_0^a v(\nu,0)\left[I_2\left(x,\frac{y}{\beta},\nu\right) - I_2(x,y,\nu)\right]d\nu
\end{aligned}\right\} \qquad (1)$$

where S_x, S_y, and S_{xy} are the stresses, which have been made dimensionless in terms of the load, $v(\nu,0)$ is the displacement of the crack (i.e., crack opening) divided by the ratio of the load to the elastic modulus in the y-direction, and

$$\beta^2 = \frac{E_y}{E_x} = \frac{\mu_y}{\mu_x} \tag{2}$$

is the orthotropy parameter. The moduli and the Poisson ratios in the y- and x-directions are E_y, E_x, μ_y, and μ_x.

The functions I_1 and I_2 are given by

$$\left.\begin{aligned} I_1\left(x,\frac{y}{\beta},\nu\right) &= \frac{1}{4}\left\{\frac{\cos \pi(\nu + x)\cosh \pi \frac{y}{\beta} - 1}{\left[\cos \pi(\nu + x) - \cosh \pi \frac{y}{\beta}\right]^2} + \frac{\cos \pi(\nu - x)\cosh \pi \frac{y}{\beta} - 1}{\left[\cos \pi(\nu - x) - \cosh \pi \frac{y}{\beta}\right]^2}\right\} \\ I_2\left(x,\frac{y}{\beta},\nu\right) &= \frac{1}{4}\left\{\frac{\sin \pi(\nu + x)\sinh \pi \frac{y}{\beta}}{\left[\cos \pi(\nu + x) - \cosh \pi \frac{y}{\beta}\right]^2} - \frac{\sin \pi(\nu - x)\sinh \pi \frac{y}{\beta}}{\left[\cos \pi(\nu - x) - \cosh \pi \frac{y}{\beta}\right]^2}\right\} \end{aligned}\right\} \tag{3}$$

and the crack opening $v(x,0)$ is given by the solution of the integral equation

$$v(x,0) = \frac{1 + \beta}{2} F(x) + \int_0^a v(\eta,0)K(x,\eta)d\eta \tag{4}$$

where

$$F(x) = \frac{2}{\pi} \int_a^1 \ln[2(\cos \pi x - \cos \pi\eta)]d\eta \tag{5a}$$

$$= \frac{4}{\pi^2} \sum_1^\infty \frac{1}{m^2} \sin m\pi a \cos m\pi x \tag{5b}$$

$$\begin{aligned} &= - \frac{4a}{\pi} (\ln \pi - 1) - \frac{2}{\pi} (a + x)\left[\ln(a + x) + \sum_{m=1}^\infty \frac{(-1)^m B_{2m}\pi^{2m}(a + x)^{2m}}{2m(2m + 1)!}\right] \\ &\quad - \frac{2}{\pi} (a - x)\left[\ln(a - x) + \sum_{m=1}^\infty \frac{(-1)^m B_{2m}\pi^{2m}(a - x)^{2m}}{2m(2m + 1)!}\right] \end{aligned} \tag{5c}$$

and

$$K(x,\eta) = a + \frac{\sin \pi a}{2\pi \sin \frac{\pi}{2}(a + \eta)\sin \frac{\pi}{2}(a - \eta)} \ln\left[4 \sin \frac{\pi}{2}(a + x)\sin \frac{\pi}{2}(a - x)\right]$$

$$- \frac{\sin \pi\eta}{2\pi \sin \frac{\pi}{2}(x + \eta)\sin \frac{\pi}{2}(x - \eta)} \ln \frac{\sin \frac{\pi}{2}(a + \eta)}{\sin \frac{\pi}{2}(a - \eta)}$$

$$+ \frac{\sin \pi x}{2\pi \sin \frac{\pi}{2}(x + \eta)\sin \frac{\pi}{2}(x - \eta)} \ln \frac{\sin \frac{\pi}{2}(a + x)}{\sin \frac{\pi}{2}(a - x)} \tag{6}$$

The procedure for obtaining the stress distribution is then to first solve equation (4) for the crack opening $v(x,0)$. This is a Fredholm integral equation of the second kind that can be solved by any standard method. A numerical successive approximation technique is used herein. Although the kernel is singular at $\eta = a$, the singularity can readily be cancelled out, and the principal value obtained as shown in appendix B. The kernel is not singular at $x = \eta$. The function $F(x)$ is a known function and can be computed by any of equations (5). The series in equation (5c) are very rapidly convergent for small a. The B_{2m} are the Bernoulli numbers 1/6, -1/30, 1/42, and so forth.

Once the solution for $v(x,0)$ has been obtained, the complete stress field can be computed by simple quadrature with equations (1). For an isotropic plate $\beta = 1$, equations (1) become, after a limiting process,

$$S_y(x,y) = 1 - \pi \int_0^a v(\nu,0)\left(I_1 - \frac{\partial I_1}{\partial y} y\right)d\nu \tag{7a}$$

$$S_x(x,y) = -\pi \int_0^a v(\nu,0)\left(I_1 + \frac{\partial I_1}{\partial y} y\right)d\nu \tag{7b}$$

$$S_{xy}(x,y) = \pi y \int_0^a v(\nu,0) \frac{\partial I_2}{\partial y} d\nu \tag{7c}$$

Equation (7c) can also be written as

$$S_{xy} = \pi y \int_0^a v(\nu,0) \frac{\partial I_1}{\partial x} d\nu \tag{8}$$

In equations (7) and (8) I_1 and I_2 are evaluated with β set equal to 1.

Isotropic Plate

Figure 1 shows the solution of equation (4) for an isotropic material. The ratio of crack opening to maximum crack opening as obtained by a successive approximation solution of equation (4) has been plotted against the ratio of the distance along the crack to the crack length. This has been done for various ratios of crack length to plate width a, and the results all fall on the quadrant of the circle shown in figure 1. This indicates that the shape of the crack remains elliptic at least up to values of a equal to 0.5. The ratio of minor to major axes increases as a increases, but this increase is only 12.5 percent in going from a = 0 to a = 0.5.

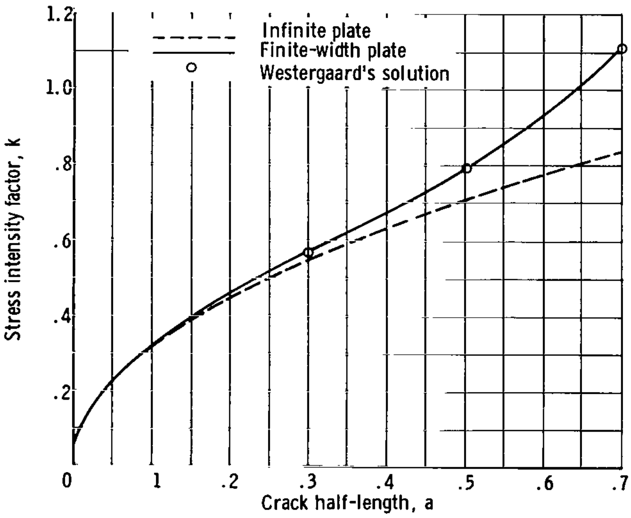

Figure 2. - Variation of stress intensity factor with crack length.

After the crack opening $v(x,0)$ is computed, the stresses were easily calculated from equations (7). Figure 2 shows the stress intensity factor as obtained in this way. The stress intensity factor is defined by

$$k = \lim_{x \to a} \left[\sqrt{2(x - a)}\, S_y(x,0) \right] \tag{9}$$

For an infinite-width plate this limit becomes $\sqrt{a}$ and is shown as the dashed curve in figure 2. The solid curve of figure 2 was obtained for a finite plate 2 units wide by using equations (7).

Orthotropic Plate

Some useful results for the orthotropic plate can be obtained directly from the results for the isotropic plate. Examination of equation (4) shows immediately that the crack opening $v(x,0)$ for an orthotropic plate differs from the crack opening of the isotropic plate by a numerical factor; that is,

$$v(x,0)_{orth} = \frac{1 + \beta}{2} v(x,0)_{iso} \tag{10}$$

Figure 1 is therefore also valid for the orthotropic plate. Furthermore, making use of this result in equations (1) shows that at $y = 0$

$$\left.\begin{aligned} S_y(x,0)_{orth} &= S_y(x,0)_{iso} \\ S_x(x,0)_{orth} &= \frac{1}{\beta} S_y(x,0)_{iso} \end{aligned}\right\} \tag{11}$$

It follows from equations (11) that

$$k_{orth} = k_{iso} \tag{12}$$

that is, the stress intensity factor is independent of the orthotropy of the material (see ref. 3).

Limitations of Solution

The desired solution for the stress distribution around a crack in a finite-width orthotropic plate under simple tension must satisfy the following equations:

Equilibrium:

$$\frac{\partial S_x}{\partial x} + \frac{\partial S_{xy}}{\partial y} = 0 \tag{13a}$$

$$\frac{\partial S_y}{\partial y} + \frac{\partial S_{xy}}{\partial x} = 0 \tag{13b}$$

Compatibility:

$$\frac{\partial^2 e_y}{\partial x^2} + \frac{\partial^2 e_x}{\partial y^2} = \frac{\partial^2 e_{xy}}{\partial x \, \partial y} \tag{14}$$

Stress-strain relations:

$$e_x = \beta^2 S_x - \mu_y S_y \tag{15a}$$

$$e_y = S_y - \mu_y S_x \tag{15b}$$

$$e_{xy} = 2\left(\frac{1 + \beta^2}{2} + \mu_y\right) S_{xy} \tag{15c}$$

The strains appearing in equations (14) and (15) are the actual strains divided by the ratio of the load P to the modulus in the y-direction E_y.

Boundary conditions:

$$\left.\begin{aligned}
S_y(x,\infty) &= 1 \\
S_x(x,\infty) &= S_{xy}(x,\infty) = 0 \\
S_y(x,0) &= 0 \qquad 0 \leq x < a \\
v(x,0) &= 0 \qquad a \leq x \leq 1 \\
S_{xy}(x,0) &= S_{xy}(1,y) = S_x(1,y) = 0 \\
S_{xy}(0,y) &= \frac{\partial S_y(0,y)}{\partial x} = \frac{\partial S_x(0,y)}{\partial x} = 0
\end{aligned}\right\} \tag{16}$$

Because of symmetry only one quadrant of the plate is considered. When the stress-strain relations of equations (15) are substituted into equation (14) and equations (13) are used, the compatibility equation is obtained in terms of stresses:

$$\frac{1 + \beta^2}{2}\left(\frac{\partial^2 S_x}{\partial x^2} + \frac{\partial^2 S_y}{\partial y^2}\right) + \beta^2 \frac{\partial^2 S_x}{\partial y^2} + \frac{\partial^2 S_y}{\partial x^2} = 0 \tag{17}$$

It can be shown by direct differentiation that the functions $I_1\left(x,\frac{y}{\beta},\nu\right)$ and $I_2\left(x,\frac{y}{\beta},\nu\right)$ satisfy the Cauchy-Riemann equations

$$\left.\begin{aligned}
\frac{\partial I_1}{\partial x} &= \frac{\partial I_2}{\partial(y/\beta)} \\
\frac{\partial I_1}{\partial(y/\beta)} &= - \frac{\partial I_2}{\partial x}
\end{aligned}\right\} \tag{18}$$

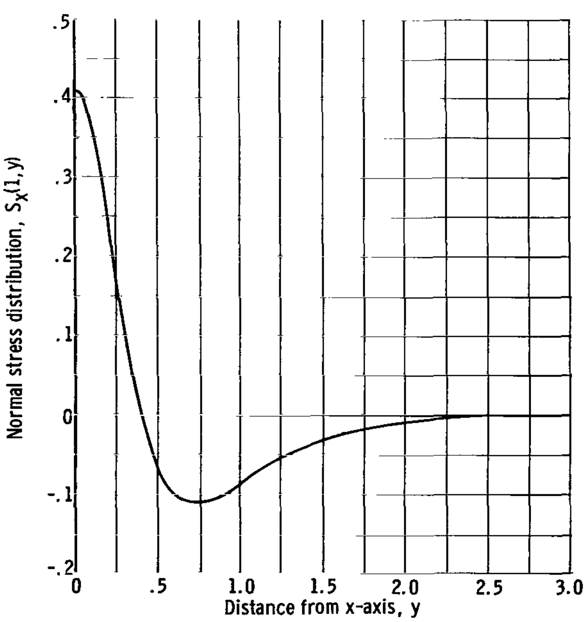

Figure 3. - Normal stress distribution on the surface $x = 1.0$, $a = 0.5$.

The solution given by equations (1) then satisfies equations (13) and (17) identically.

When the boundary conditions given by equations (16) are considered, it can be shown that equations (1) satisfy all these conditions but one. The condition that the normal stress $S_x(1,y)$ on the free surface $x = 1$ be zero is not satisfied. Instead, a normal stress distribution such as that shown in figure 3 for an isotropic plate with $a = 0.5$ is obtained. The resultant stress, however, vanishes, as can be shown by integration. By Saint-Venant's principle it would seem that the error in neglecting the effect of this normal stress distribution on the stress field in the vicinity of the crack should not be large, if the ratio of crack length to plate width is less than 1/2. This error, of course, decreases as a decreases. This is in agreement with the conclusion of reference 1.

The existence of the normal stress distribution $S_x(1,y)$ arises from the fact that solution (1) implies a constant normal displacement $u(1,y)$ at this surface. This is shown in appendix A in the derivation of equations (1). The normal stress distribution on this surface is the one required to keep this boundary straight. It follows, therefore, that solution (1) should be the same as the solution for an infinite set of colinear cracks. For such a set, the plane midway between two cracks suffers no displacement, so the solution presented satisfies all the necessary boundary conditions.

For an infinite set of colinear cracks there is available a solution for an isotropic plate in terms of complex potentials obtained by Westergaard's semi-inverse method (ref. 2). The values obtained for the stress intensity factor by using Westergaard's equations are shown by the circles in figure 2. The agreement with the present solution is excellent, as would be expected. It was also found that the crack opening as obtained from Westergaard's equations satisfies equation (4). Westergaard's solution is, of course, not valid for an orthotropic plate.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, December 10, 1963

APPENDIX A

DERIVATION

The solution equations (1) can be obtained in a formal manner by the use of finite Fourier transforms. When a Fourier sine transform is performed on equation (13a) and Fourier cosine transforms on equations (13b) and (17), there result the following three equations for the transformed stresses:

$$\left.\begin{aligned}
&\frac{dS_{xy,s}}{dy} - m\pi S_{x,c} = 0 \\
&\frac{dS_{y,c}}{dy} + m\pi S_{xy,s} = 0 \\
&\frac{1+\beta^2}{2}\left(\frac{d^2S_{y,c}}{dy^2} - m^2\pi^2 S_{x,c}\right) + \beta^2\,\frac{d^2S_{x,c}}{dy^2} - m^2\pi^2 S_{y,c} = -(-1)^m \left.\frac{\partial S_y}{\partial x}\right|_{x=1}
\end{aligned}\right\} \tag{A1}$$

where some of the conditions of equations (16) have been used and the transformed stresses are defined by

$$\left.\begin{aligned}
S_{xy,s}(m,y) &\equiv \int_0^1 S_{xy}\sin m\pi x\,dx \\
S_{x,c}(m,y) &\equiv \int_0^1 S_x \cos m\pi x\,dx,\ \text{etc.}
\end{aligned}\right\} \tag{A2}$$

Equations (A1) are ordinary differential equations for the transformed stresses, and their solutions can be written in terms of exponentials:

$$\left.\begin{aligned}
S_{y,c} &= A_m e^{-m\pi y/\beta} + B_m e^{-m\pi y} + S_P(m,y) \\
S_{x,c} &= -\frac{1}{\beta^2} A_m e^{-m\pi y/\beta} - B_m e^{-m\pi y} - \frac{1}{m^2\pi^2}\frac{d^2S_P}{dy^2} \\
S_{xy,s} &= \frac{1}{\beta} A_m e^{-m\pi y/\beta} + B_m e^{-m\pi y} - \frac{1}{m\pi}\frac{dS_P}{dy}
\end{aligned}\right\} \tag{A3}$$

where $S_P(m,y)$ is a particular solution as yet unknown and A_m and B_m are constants to be determined. The general solution of equations (A1) actually contains terms with positive exponentials as well, but these must vanish if the stresses are to remain finite as y approaches infinity.

Taking the inverse transform (ref. 4) results in

$$S_y(x,y) = S_{y,c}(0,y) + 2 \sum_{m=1}^{\infty} S_{y,c} \cos m\pi x \tag{A4a}$$

$$S_x(x,y) = S_{x,c}(0,y) + 2 \sum_{m=1}^{\infty} S_{x,c} \cos m\pi x \tag{A4b}$$

$$S_{xy}(x,y) = 2 \sum_{m=1}^{\infty} S_{xy,s} \sin m\pi x \tag{A4c}$$

From equation (A4a) it follows that

$$\left.\frac{\partial S_y}{\partial x}\right|_{x=1} = 0$$

and consequently that S_P is identically zero. It can also be shown that in order to satisfy the compatibility equation (eq. (17)) and the condition of zero S_x at infinity, $S_{x,c}(0,y)$ must vanish. Substituting equations (A3) into equations (A4) and making use of the condition

$$S_{y,c}(0,y) = \int_0^1 S_y \, dx = 1 \tag{A5}$$

result in

$$\left.\begin{aligned}
S_y(x,y) &= 1 + 2 \sum_{m=1}^{\infty} \left(A_m e^{-m\pi y/\beta} + B_m e^{-m\pi y}\right) \cos m\pi x \\
S_x(x,y) &= -2 \sum_{m=1}^{\infty} \left(\frac{1}{\beta^2} A_m e^{-m\pi y/\beta} + B_m e^{-m\pi y}\right) \cos m\pi x \\
S_{xy}(x,y) &= 2 \sum_{m=1}^{\infty} \left(\frac{1}{\beta} A_m e^{-m\pi y/\beta} + B_m e^{-m\pi y}\right) \sin m\pi x
\end{aligned}\right\} \tag{A6}$$

The condition $S_{xy}(x,0) = 0$ will be satisfied if it is assumed that

$$\frac{1}{\beta} A_m + B_m = 0 \tag{A7}$$

Equations (A6) then become

$$\left.\begin{aligned}
S_y(x,y) &= 1 + 2\sum_{m=1}^{\infty} A_m\left(e^{-m\pi y/\beta} - \frac{1}{\beta}\, e^{-m\pi y}\right)\cos m\pi x \\
S_x(x,y) &= -2\sum_{m=1}^{\infty} \frac{A_m}{\beta^2}\left(e^{-m\pi y/\beta} - \beta e^{-m\pi y}\right)\cos m\pi x \\
S_{xy}(x,y) &= 2\sum_{m=1}^{\infty} \frac{A_m}{\beta}\left(e^{-m\pi y/\beta} - e^{-m\pi y}\right)\sin m\pi x
\end{aligned}\right\} \tag{A8}$$

It should be noted that equations (A8) could also have been obtained by assuming for the stresses Fourier series whose coefficients are functions of y and then determining these functions by substituting into the equilibrium and the compatibility equations.

To obtain the displacements, a Fourier cosine transform is performed on equation (15a), while a Fourier sine transform is performed on equation (15c). After some algebraic manipulation, making use of equations (A3) and (A7) and then taking the inverse transforms, there result

$$v(x,y) = \int_0^1 v(x,y)dx - \frac{2}{\beta}\sum_{m=1}^{\infty} \frac{A_m}{m\pi}\left[\left(\beta^2 + \mu_y\right)e^{-m\pi y/\beta} - (1 + \mu_y)e^{-m\pi y}\right]\cos m\pi x \tag{A9a}$$

$$u(x,y) = -2\sum_{m=1}^{\infty} \frac{A_m}{m\pi}\left[(1 + \mu_y)e^{-m\pi y/\beta} - \left(\beta + \frac{\mu_y}{\beta}\right)e^{-m\pi y}\right]\sin m\pi x - \mu_y x \tag{A9b}$$

where equation (15a) has been used to determine $u(1,y) = -\mu_y$. From equation (A9b) it follows that the solution implies a uniform normal displacement $-\mu_y$ at the edge of the plate.

The coefficients A_m can now be determined from the conditions at $y = 0$. Thus, at $y = 0$,

$$S_y(x,0) = 1 + 2\sum_{m=1}^{\infty} A_m\left(1 - \frac{1}{\beta}\right)\cos m\pi x \tag{A10a}$$

$$v(x,0) = \int_0^a v(x,0)dx + 2\,\frac{1 - \beta^2}{\beta}\sum_{m=1}^{\infty}\frac{A_m}{m\pi}\cos m\pi x \tag{A10b}$$

where the condition $v(x,0) = 0$, $a \leq x \leq 1$, has been used. Equations (A10) are merely Fourier series, and it consequently follows that

$$A_m\left(1 - \frac{1}{\beta}\right) = \int_a^1 S_y(\zeta,0)\cos m\pi\zeta\; d\zeta \tag{A11a}$$

$$\frac{1 - \beta^2}{\beta}\,\frac{A_m}{m\pi} = \int_0^a v(\nu,0)\cos m\pi\nu\; d\nu \tag{A11b}$$

where the conditions $S_y(x,0) = 0$, $0 \leq x < a$, and $v(x,0) = 0$, $a \leq x \leq 1$, have been used.

Substituting equation (A11a) into equation (A10b) and equation (A11b) into equation (A10a) yields (after orders of summation and integration are reversed)

$$S_y(\zeta,0) = 1 - \frac{2}{1 + \beta}\int_0^a v(\nu,0)\left(\sum_{m=1}^{\infty} m\pi\cos m\pi\nu\cos m\pi\zeta\right)d\nu \tag{A12a}$$

$$v(x,0) = \int_0^a v(x,0)dx - 2(1 + \beta)\int_a^1 S_y(\zeta,0)\left(\sum_{m=1}^{\infty}\frac{1}{m\pi}\cos m\pi\zeta\cos m\pi x\right)d\zeta \tag{A12b}$$

The divergent series appearing in equations (A12) can be formally summed (ref. 5):

$$\left.\begin{aligned}
\sum_{m=1}^{\infty} m\pi \cos m\pi\nu \cos m\pi\zeta &= -\frac{\pi}{2}\,\frac{1 - \cos \pi\nu \cos \pi\zeta}{(\cos \pi\nu - \cos \pi\zeta)^2} \\
\sum_{m=1}^{\infty} \frac{1}{m\pi} \cos m\pi\zeta \cos m\pi x &= -\frac{1}{4\pi} \log\left[4(\cos \pi x - \cos \pi\zeta)^2\right]
\end{aligned}\right\} \qquad \text{(A13)}$$

If equations (A13) are substituted into equations (A12) and equation (A12a) is substituted into equation (A12b), after the order of integration is reversed and one integration is performed, the integral equation (4) is obtained. To obtain the stresses (eqs. (1)), equation (A11b) is substituted into equations (A8) and the resultant series summed, as was done previously, after the order of summation and integration is reversed. The correctness of the solution can be determined by substituting back into the equilibrium and the compatibility equations.

APPENDIX B

EVALUATION OF SINGULAR INTEGRAL

The integral in equation (4) is singular at $\eta = a$ since the kernel is unbounded at this point. Since the kernel $K(x,\eta)$ goes to infinity as $1/\epsilon$ and the displacement $v(\eta,0)$ goes to zero as $\epsilon^{1/2}$, the integral has a square root singularity at $\eta = a$. The principal value can be calculated as follows. Equation (4) can be written as

$$v(x,0) = \frac{1 + \beta}{2} F(x) + \int_0^{a-\epsilon} v(\eta,0)K(x,\eta)d\eta + \int_{a-\epsilon}^{a} v(\eta,0)K(x,\eta)d\eta \tag{B1}$$

where ϵ is a small positive number. The first integral is nonsingular and can be evaluated without difficulty. The second integral, which for convenience will be designated by J, can be written as

$$J = \int_{a-\epsilon}^{a} \left[v(\eta,0)K(x,\eta) - C \frac{f(x,a)}{\left(a^2 - \eta^2\right)^{1/2}}\right]d\eta + C \int_{a-\epsilon}^{a} \frac{f(x,a)}{\left(a^2 - \eta^2\right)^{1/2}} d\eta \tag{B2}$$

where

$$\begin{aligned} f(x,a) &= \lim_{\eta \to a}\left[(a^2 - \eta^2)K(x,\eta)\right] \\ &= \frac{2a}{\pi^2} \ln\left(4 \sin \pi \frac{a + x}{2} \sin \pi \frac{a - x}{2}\right) \end{aligned} \tag{B3}$$

and C is a constant. The integrand of the first integral in (B2) is now bounded at $\eta = a$, and the second integral can be readily evaluated. Thus,

$$\int_{a-\epsilon}^{a} \frac{f(x,a)}{\left(a^2 - \eta^2\right)^{1/2}} d\eta = \frac{2a}{\pi^2} \ln\left(4 \sin \pi \frac{a + x}{2} \sin \pi \frac{a - x}{2}\right)\sin^{-1}\left(\frac{2\epsilon}{a} - \frac{\epsilon^2}{a^2}\right)^{1/2} \tag{B4}$$

Since $v(\eta,0)$ is elliptic in the neighborhood of a, the first integral in equation (B2) can be written with negligible error as

$$C = \int_{a-\epsilon}^{a} \sqrt{a^2 - \eta^2} \left[K(x,\eta) - \frac{f(x,a)}{a^2 - \eta^2}\right] d\eta$$

and the condition of continuity for v at $\eta = a - \epsilon$ leads to

$$C = \frac{v(a - \epsilon,0)}{\sqrt{2a\epsilon - \epsilon^2}} \tag{B5}$$

The equation for J becomes

$$J = C\left\{\int_{a-\epsilon}^{a} \sqrt{a^2 - \eta^2} \left[K(x,\eta) - \frac{f(x,a)}{a^2 - \eta^2}\right] d\eta + f(x,a)\sin^{-1}\left(\frac{2\epsilon}{a} - \frac{\epsilon^2}{a^2}\right)^{1/2}\right\} \tag{B6}$$

where $f(x,a)$ is given by equation (B3) and C by equation (B5). For the calculations performed herein, ϵ was always chosen to be $a/100$.

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